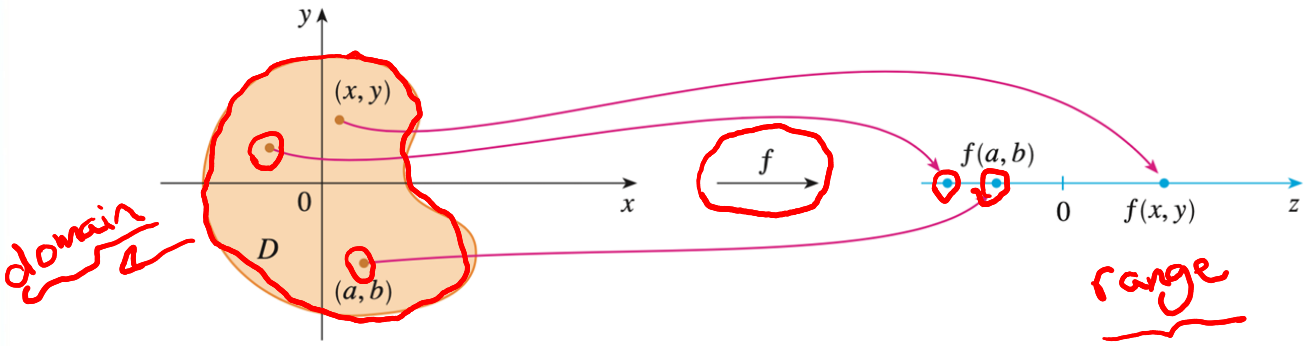
# 14.1 Functions of Several Variables

domain

**Function of Two Variables**

A **function** $f$ of two variables is a rule that assigns to each ordered pair of real numbers $(x, y)$ in a set $D \subset \mathbb{R}^2$ a unique real number denoted by $f(x, y)$. The set $D$ is the **domain** of $f$ and its **range** is the set of values that $f$ takes on. We often write $z = f(x, y)$.

range




domain

range

Ex

$$f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$$

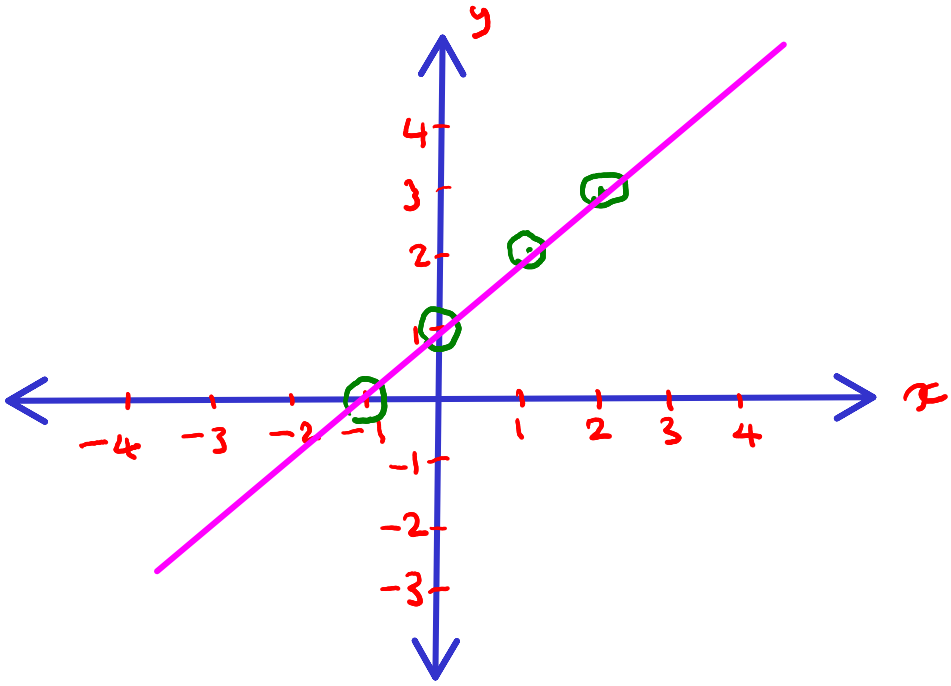
find $f(\overset{x}{2}, \overset{y}{3})$

$$f(2,3) = \frac{\sqrt{2+3+1}}{2-1} = \sqrt{6}$$

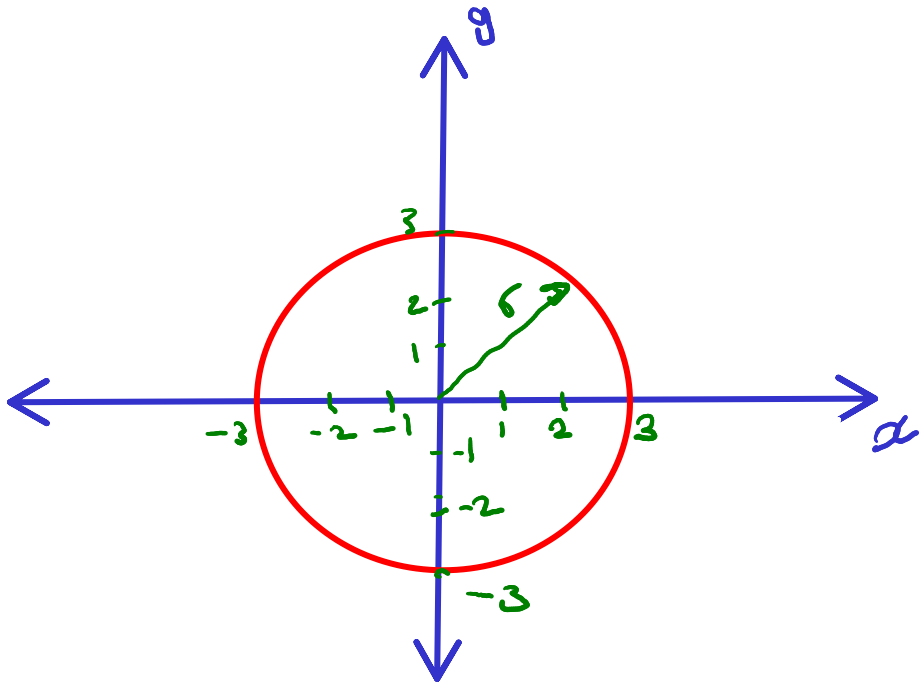
# Line

$$y = x + 1$$

| x | -1 | 0 | 1 | 2 |
|---|---|---|---|---|
| y | 0 | 1 | 2 | 3 |



# Circle




$$x^2 + y^2 = r^2$$ ← radius نصف القطر

$$D = 2r$$

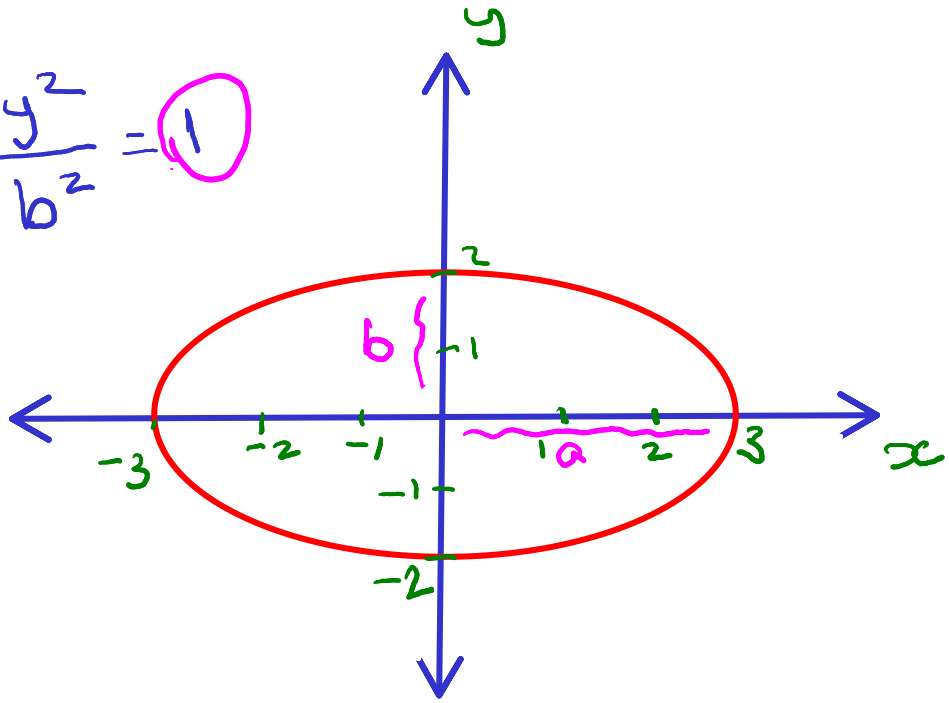
Ex $x^2 + y^2 = 9$

$$\sqrt{r^2} = \sqrt{9}$$

$$\boxed{r = 3}$$

# Ellispe : قطع ناقص

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$




$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$a^2 = 9 \qquad a = \sqrt{9} = 3$

$b^2 = 4 \qquad b = \sqrt{4} = 2$

## Example 1

For the following function, evaluate $f(3,2)$ and find and sketch the domain.

$$f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$$

إذن

$x+y+1 \geq 0$

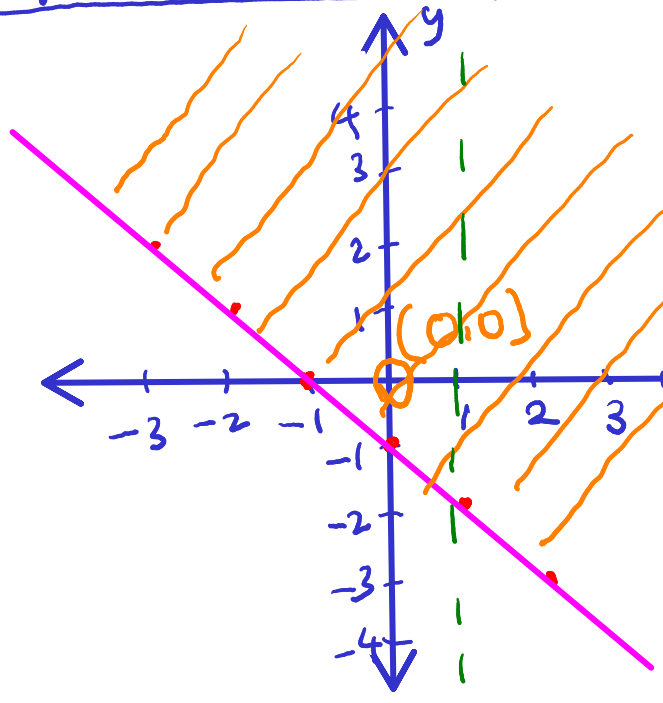
$y \geq -x-1$

لكن

$x-1 \neq 0$

$x \neq 1$

| x | -3 | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|---|
| y | 2 | 1 | 0 | -1 | -2 | -3 |




$y = -x-1$

$y \geq -x-1$

$0 \geq 0-1$

$0 \geq -1$

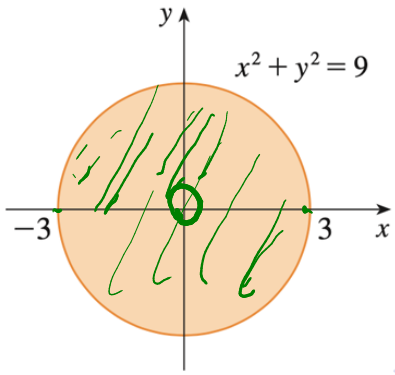
**Example 2**

Find the domain of $g(x,y) = \sqrt{9 - x^2 - y^2}$.

The quantity under the square root should be greater or equal to 0 $(9 - x^2 - y^2 \geq 0)$. So the domain of $g$ is:

$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9\}$$

We first draw the circle $x^2 + y^2 = 0$:




$x^2+y^2 \leq 9$

$0^2+0^2 \leq 9$ ✓

$$f(x) = \sqrt{9-x^2-y^2}$$

$$9-x^2-y^2 \geq 0$$

$$-x^2-y^2 \geq -9 \quad *-1$$

$$x^2+y^2 \leq 9$$

$$x^2+y^2 = 9$$

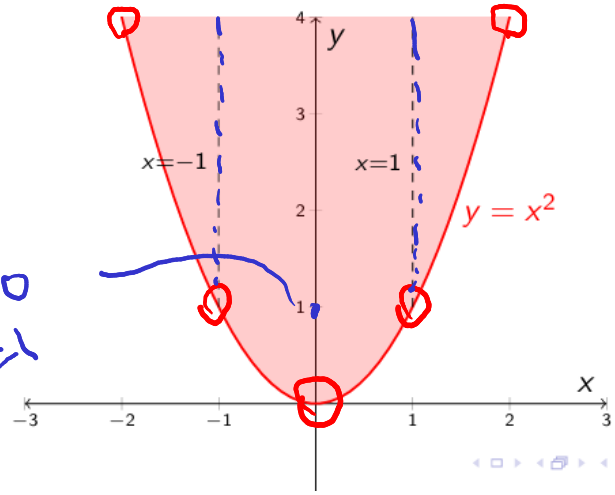
$$r^2 = 9$$

$$\boxed{r=3}$$

**Example 3**

Find the domain of $f(x,y) = \dfrac{\sqrt{y-x^2}}{1-x^2}$.

$D = \{(x,y) \in \mathbb{R}^2 \mid y \geq x^2,\ x \neq -1,\ x \neq 1\}$




$x=0$
$y=1$

$\sqrt{y-x^2}$
$y-x^2 \geq 0$
$1-0^2 \geq 0$

$f(x) = \dfrac{\sqrt{y-x^2}}{1-x^2}$

$y-x^2 \geq 0$

$y-x^2 = 0$

$y = x^2$ ⇝ parabola

| −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|
| 4 | 1 | 0 | 1 | 4 |

$1-x^2 \neq 0$

$x^2 \neq 1$

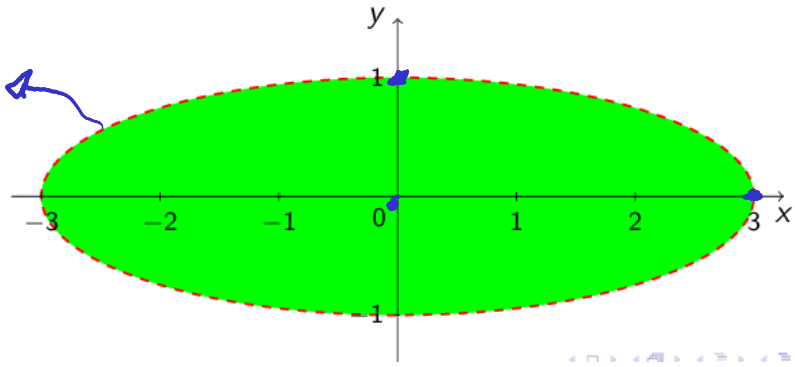
$x \neq -1, 1$

**Example 4**

Find the domain of $f(x,y) = \ln\left(9 - x^2 - 9y^2\right)$.

$$9 - x^2 - 9y^2 > 0 \iff x^2 + 9y^2 < 9 \iff \boxed{\frac{x^2}{9} + y^2 < 1}$$

$$D = \{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{9} + y^2 < 1\}$$




$\frac{x^2}{9} + y^2 < 1$

$\frac{0^2}{9} + 0^2 < 1$

$9 - x^2 - 9y^2 > 0$

$-x^2 - 9y^2 > -9 \quad \div -1$

$x^2 + 9y^2 > 9$

$\frac{x^2}{9} + \frac{9y^2}{9} = \frac{9}{9}$

$\frac{x^2}{9} + \frac{y^2}{1} = 1$

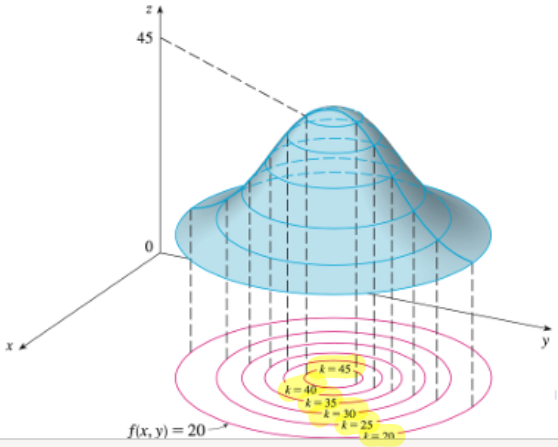
$a^2 \leftarrow 9 \qquad 1 \rightarrow b^2$

$a = \sqrt{9} = 3 \qquad b = \sqrt{1} = 1$

## Level Curves

Another method for visualizing a function is a contour map of which points of constant elevation are joined to form **contour curves** or **level curves**.

The **level curves** of a function $f$ of two variables are the curves with equation $f(x, y) = k$, where $k$ is a constant (in the range of $f$).






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**Example 6**

Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

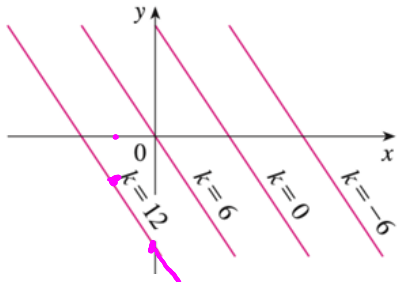
For $k = -6$, we get: $6 - 3x - 2y = -6 \iff \boxed{3x + 2y - 12 = 0}$

For $k = 0$, we get: $6 - 3x - 2y = 0 \iff \boxed{3x + 2y - 6 = 0}$

For $k = 6$, we get: $6 - 3x - 2y = 6 \iff \boxed{3x + 2y = 0}$

For $k = 12$, we get: $6 - 3x - 2y = 12 \iff \boxed{3x + 2y + 6 = 0}$

**This is a family of lines**




$f(x,y) = 6 - 3x - 2y = k$

$k = -6$

$6 - 3x - 2y = -6$

$-3x - 2y = -6 + 6 = 0$

$\boxed{-3x - 2y = 0}$

$k = 0$ $\quad f(x,y) = 6 - 3x - 2y = 0$

$\boxed{-3x - 2y = -6}$

$k = 6$ $\quad -6 - 3x - 2y = 6$

$-3x - 2y = 6 + 6 =$

$\boxed{-3x - 2y = 12}$

$k = 12$

$6 - 3x - 2y = 12$

$-3x - 2y = 12 - 6$

$-3x - 2y = 6$

| $x$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $y$ | $-1.5$ | $-3$ | $-4.5$ |

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**Example 7**

Sketch the level curves of the function $g(x,y) = \sqrt{9 - x^2 - y^2}$ for the values $k = 0, 1, 2, 3$.

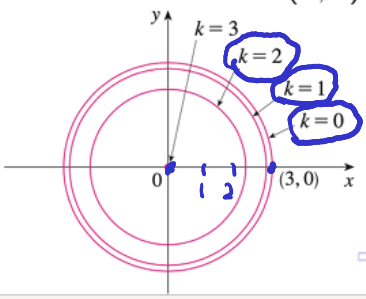
For $k = 0$, we get: $\sqrt{9 - x^2 - y^2} = 0 \iff \boxed{x^2 + y^2 = 9}$

For $k = 1$, we get: $\sqrt{9 - x^2 - y^2} = 1 \iff \boxed{x^2 + y^2 = 8}$

For $k = 2$, we get: $\sqrt{9 - x^2 - y^2} = 2 \iff \boxed{x^2 + y^2 = 5}$

For $k = 3$, we get: $\sqrt{9 - x^2 - y^2} = 3 \iff \boxed{x^2 + y^2 = 0}$

**This is a family of circles with center $(0, 0)$**




10

$$f(x,y) = \sqrt{9 - x^2 - y^2}$$

$k = 0$

$$\sqrt{9 - x^2 - y^2} = 0$$

$$9 - x^2 - y^2 = 0$$

$$-x^2 - y^2 = -9 \qquad \div -1$$

$$x^2 + y^2 = 9$$

$$r = \sqrt{9} = 3$$

$k=1$

$$\sqrt{9-x^2-y^2} = 1$$

$$9-x^2-y^2 = 1$$

$$-x^2-y^2 = 1-9 = -8$$

$$x^2+y^2 = 8$$

$$r = \sqrt{8} = 2.8$$

---

$k=2$

$$\sqrt{9-x^2-y^2} = 2$$

$$9-x^2-y^2 = 4$$

$$-x^2-y^2 = 4-9 = -5$$

$$x^2+y^2 = 5$$

$$r = \sqrt{5} = 2.2$$

$k=3$

$$\sqrt{9-x^2-y^2}=3$$

$$9-x^2-y^2=9$$

$$-x^2-y^2=0$$

$$x^2+y^2=0$$

$$\boxed{r=0}$$

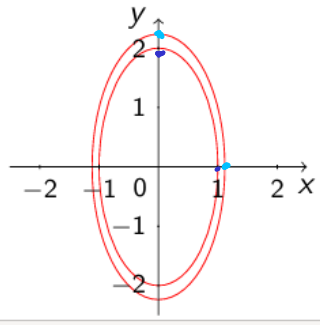
**Example 8**

Sketch the level curves of the function $h(x, y) = 4x^2 + y^2 + 1$ for the values $k = 5, 6$.

$$k = 5 : 4x^2 + y^2 + 1 = 5 \iff 4x^2 + y^2 = 4 \iff \boxed{x^2 + \frac{y^2}{4} = 1}$$

$$k = 6 : 4x^2 + y^2 + 1 = 6 \iff 4x^2 + y^2 = 5 \iff \boxed{\frac{x^2}{5/4} + \frac{y^2}{5} = 1}$$

**This is a family of ellipses**




11 / 1

$f(x) = 4x^2 + y^2 + 1 = k$

$\boxed{k = 5}$

$4x^2 + y^2 + 1 = 5$

$\frac{4}{4}x^2 + \frac{y^2}{4} = \frac{4}{4}$

$\frac{x^2}{1} + \frac{y^2}{4} = 1$

$a = \sqrt{1} = 1$ $\quad b = \sqrt{4} = 2$

$k = 6$

$$4x^2 + y^2 + 1 = 6$$

$$\frac{4x^2}{5} + \frac{y^2}{5} = 6 - 1 = \frac{5}{5}$$

$$\frac{x^2}{\frac{5}{4}} + \frac{y^2}{5} = 1$$

$$a^2 = \frac{5}{4} \qquad b^2 = 5$$

$$a = \sqrt{\frac{5}{4}} = 1.11 \qquad b = \sqrt{5} = 2.23$$