Ma102-7.1

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7.1 (2)

7.1 Integration by Parts

Formula

The formula for **integration by parts** is:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

or:

$$\int u\,dv = uv - \int v\,du$$

How We Choose u

1. Logarithmic function — ln, log
2. Inverse Trigonometric function — $\tan^{-1}$, $\sin^{-1}$, $\cos^{-1}$
3. Polynomial or power function — x, x^2, x^3, x^4 ...
4. Trigonometric function — sin, cos, tan
5. Exponential function — e^x

u ↓ قابلة للاشتقاق — dv ↓ تكامل

du — v

7.1 Integration by Parts

Example 1

Evaluate the integral $\int x \sin x\,dx$

Solution

Let: $u = x \longrightarrow du = dx$

and: $dv = \sin x dx \longrightarrow v = \int \sin x dx = -\cos x$

$$\int x \sin x\,dx = uv - \int v\,du$$

$$= -x\cos x - \int -\cos x\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

$\int x \sin x\,dx$

$u = x$ — $du = \sin x\,dx$

$du = 1\,dx$ — $v = \int \sin x\,dx$

$v = -\cos x$

$uv - \int v\,du$

$\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx$

$= -x\cos x + \int \cos x\,dx$

$= -x\cos x + \sin x + c$

7.1 Integration by Parts

Example 2

Evaluate the integral $\int \ln x\,dx$

Solution

Let: $u = \ln x \Longrightarrow du = \frac{1}{x}dx$

and: $dv = dx \longrightarrow v = \int dx = x$

$$\int \ln x\,dx = uv - \int v\,du$$

$$= x\ln x - \int x \cdot \frac{1}{x}dx$$

$$= x\ln x - x + C$$

$\int \ln x\,dx$

$u = \ln x$ — $dv = 1dx$

$du = \frac{1}{x}dx$ — $v = x$

$\int \ln x\,dx = x\ln x - \int x \cdot \frac{1}{x}dx$

$= x\ln x - \int 1\,dx$

$= x\ln x - x + c$

7.1 Integration by Parts

Example 3

Evaluate the integral $\int t^2 e^t\,dt$

Solution

Let: $u = t^2 \longrightarrow du = 2tdt$

and: $dv = e^t dt \longrightarrow v = \int e^t dt = e^t$

$$\int t^2 e^t dt = uv - \int v\,du = t^2 e^t - 2\int te^t dt$$

Again Integration by Parts:

$u = t \longrightarrow du = dt$

$dv = e^t dt \longrightarrow v = \int e^t dt = e^t$

$\int t^2 e^t dt$

$u = t^2$ — $dv = e^t dt$

$du = 2t\,dt$ — $v = e^t$

$\int t^2 e^t dt = t^2 e^t - \int 2te^t dt$

$= t^2 e^t - 2\int te^t dt$

7.1 Integration by Parts

$$\int t^2 e^t dt = t^2 e^t - 2\left(uv - \int v\,du\right)$$

$$= t^2 e^t - 2\left(te^t - \int e^t dt\right)$$

$$= t^2 e^t - 2(te^t - e^t) + C$$

$$= t^2 e^t - 2te^t + 2e^t + C$$

$\int te^t dt$

$u = t$ — $dv = e^t$

$du = dt$ — $v = e^t$

$= t^2e^t - 2\left(te^t - \int e^t\,dt\right)$

$= t^2e^t - 2\left(te^t - e^t\right) + C$

$= t^2e^t - 2te^t + 2e^t + C$

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7.1 Integration by Parts

Example 4

Evaluate the integral $\int \frac{\ln x}{x^5}\,dx$

Solution

Let: $u = \ln x \implies du = \frac{1}{x}\,dx$

and: $dv = \frac{dx}{x^5} \implies v = \int x^{-5}\,dx = \frac{x^{-4}}{-4}$

$$\int \frac{\ln x}{x^5}\,dx = uv - \int v\,du$$

$$= -\frac{\ln x}{4x^4} - \int \frac{x^{-4}}{-4}\cdot\frac{1}{x}\,dx = -\frac{\ln x}{4x^4} + \frac{1}{4}\int x^{-5}\,dx$$

$$= -\frac{\ln x}{4x^4} + \frac{1}{4}\cdot\frac{x^{-4}}{-4} + C = -\frac{\ln x}{4x^4} - \frac{1}{16x^4} + C$$

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7.1 Integration by Parts

Example 5

Evaluate the integral $\int \tan^{-1} x\,dx$

Solution

Let: $u = \tan^{-1} x \implies du = \frac{1}{x^2+1}dx$

and: $dv = dx \implies v = \int dx = x$

$$\int \tan^{-1} x\,dx = uv - \int v\,du = x\tan^{-1} x - \int \frac{x}{x^2+1}\,dx$$

The integral $\int \frac{x}{x^2+1}\,dx$ can be solved using the substitution rule:

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7.1 Integration by Parts

Let: $u = x^2 + 1 \implies du = 2x dx \implies dx = \frac{du}{2x}$

So $\int \frac{x}{x^2+1}\,dx = \int \frac{x}{u}\frac{du}{2x} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+1|$

Finally:

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\ln|x^2+1| + C$$

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7.1 Integration by Parts

Example 6

Evaluate the integral $\int e^x \sin x\,dx$

Solution

Let: $u = \sin x \implies du = \cos x dx$

and: $dv = e^x dx \implies v = \int e^x dx = e^x$

$$\int e^x \sin x\,dx = uv - \int v\,du = \sin x e^x - \int e^x \cos x\,dx$$

Again Integration by Parts:

Let: $u = \cos x \implies du = -\sin x dx$

and: $dv = e^x dx \implies v = \int e^x dx = e^x$

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7.1 Integration by Parts

$$\int e^x \sin x\,dx = \sin x e^x - \int e^x \cos x\,dx$$

$$= \sin x e^x - \left[\cos x e^x - \int e^x(-\sin x)\,dx\right]$$

$$= \sin x e^x - \cos x e^x - \int e^x \sin x\,dx$$

$u = t \qquad dv = e^t$

$du = dt \qquad v = e^t$

$\int te^t = te^t - \int e^t dt$

$= te^t - e^t$

$\int t^2 e^t = t^2 e^t - 2(te^t - e^t)$

$= t^2 e^t - 2te^t + 2e^t + C$

$\int \frac{\ln x}{x^5}\,dx$

$u = \ln x \qquad dv = \frac{1}{x^5}dx = x^{-5}dx$

$du = \frac{1}{x}dx \qquad v = \int x^{-5}dx = \frac{x^{-4}}{-4}$

$\int \frac{\ln x}{x^5}dx = \frac{x^{-4}\ln x}{-4} - \int \frac{x^{-4}}{-4}\cdot\frac{1}{x}dx$

$= \frac{x^{-4}\ln x}{-4} + \int \frac{x^{-5}}{4}dx$

$= \frac{x^{-4}\ln x}{-4} - \frac{x^{-4}}{16} + C$

$\int \tan^{-1} x\,dx$

$u = \tan^{-1} x \qquad du = 1\,dx$

$du = \frac{1}{x^2+1}dx \qquad v = x$

$\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{x^2+1}dx$

$= x\tan^{-1} x - \frac{1}{2}\ln|x^2+1| + C$

$\int e^x \sin x\,dx$

$u = \sin x \qquad dv = e^x dx$

$du = \cos x\,dx \qquad v = e^x$

$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$

$= \sin x e^x - \left[\cos x e^x - \int e^x(-\sin x)\,dx\right]$

$= \sin x e^x - \cos x e^x - \int e^x \sin x\,dx$

That gives:

$2\int e^x \sin x\,dx = \sin x e^x - \cos x e^x$

$\int e^x \sin x\,dx = \frac{\sin x e^x - \cos x e^x}{2}$

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7.1 Integration by Parts

Practice Questions

(a) $\int x \cos 5x\,dx$ (b) $\int t^2 \sin 4t\,dt$

(c) $\int t \sec^2 2t\,dt$ (d) $\int te^{-3t}\,dt$

(e) $\int (x^2+2x)\,e^{-x}\,dx$ (f) $\int t^4 \ln t\,dt$

(g) $\int \frac{\ln x}{x^2}\,dx$ (h) $\int \sqrt{x} \ln x\,dx$

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$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$

$\int e^x \cos x\,dx$

$u = \cos x \qquad dv = e^x dx$

$du = -\sin x\,dx \qquad v = e^x$

$\int e^x \cos x\,dx = \cos x e^x + \int e^x \sin x\,dx$

$\int e^x \sin x\,dx = e^x \sin x - \cos x e^x - \int e^x \sin x\,dx$

$\int e^x \sin x\,dx + \int e^x \sin x\,dx = e^x \sin x - \cos x e^x$

$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$

$\int e^x \sin x\,dx = \frac{e^x \sin x - e^x \cos x}{2}$

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a) $\int x \cos 5x\,dx$

$u = x \qquad dv = \cos 5x\,dx$

$du = 1\,dx \qquad v = \int \cos 5x\,dx$

$v = \frac{1}{5}\sin 5x$

$\int x \cos 5x\,dx = \frac{1}{5} x \sin 5x - \frac{1}{5}\int \sin 5x\,dx$

$= \frac{1}{5} x \sin 5x - \frac{1}{5} \cdot \frac{-\cos 5x}{5} + C$

$= \frac{1}{5} x \sin 5x + \frac{1}{25}\cos 5x + C$

$\int t \sec^2 2t\,dt$

$u = t \qquad dv = \sec^2 2t\,dt$

$u = t \qquad dv = \sec^2 2t\,dt$

$du = 1\,dt \qquad v = \int \sec^2 2t\,dt$

$v = \frac{1}{2}\tan 2t$

$$\int t\sec^2 2t\,dt = \frac{1}{2}t\tan 2t - \frac{1}{2}\int \tan 2t\,dt$$

$$= \frac{1}{2}t\tan 2t - \frac{1}{2}\ln|\sec 2t| + C$$

$$\int \frac{\ln x}{x^2}\,dx$$

$u = \ln x \qquad dv = \frac{1}{x^2}dx = x^{-2}dx$

$du = \frac{1}{x}dx \qquad v = \frac{x^{-1}}{-1}$

$$\int \frac{\ln x}{x^2}\,dx = \frac{\ln|x|\cdot x^{-1}}{-1} - \int \frac{x^{-1}}{-1}\cdot\frac{1}{x^1}\,dx$$

$$= -\frac{\ln|x|}{x} + \int x^{-2}\,dx$$

$$= \frac{-\ln|x|}{x} + \frac{x^{-1}}{-1} + C = \frac{-\ln|x|}{x} - \frac{1}{x} + C$$

$$\int \sqrt{x}\ln x\,dx$$

$u = \ln x \qquad dv = x^{\frac{1}{2}}dx$

$du = \frac{1}{x}dx \qquad v = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x^{\frac{3}{2}}$

$$\int \sqrt{x}\ln x\,dx = \frac{2}{3}\ln x\cdot x^{\frac{3}{2}} - \frac{2}{3}\int x^{\frac{3}{2}}\cdot\frac{1}{x^1}\,dx$$

$$= \frac{3}{2} - \int \frac{1}{2}\cdot$$

$$= \frac{2}{3}\ln|x| \cdot x^{\frac{3}{2}} - \frac{2}{3}\int x^{\frac{1}{2}}\,dx$$

$$= \frac{2}{3}\ln|x| \cdot x^{\frac{3}{2}} - \frac{2}{3}\cdot\frac{2}{3}x^{\frac{3}{2}} + C$$

$$= \frac{2}{3}\ln|x| \cdot x^{\frac{3}{2}} - \frac{4}{9}x^{\frac{3}{2}} + C$$